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LETTER TO THE EDITOR

General bond-to-site mapping in aggregation

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Abstract. We present an exact bond-to-site transformation for diffusion-limited aggregation. The equivalence maps a class of partial adhesion problems on arbitrary lattices to absolute adhesion problems on transformed lattices. Examples are given for diffusion in the presence and absence of an external field. The correspondence relates two conjectured aspects of universality in the scaling behaviour, namely the irrelevance of lattice structure and adhesion probability.

Since the introduction of the Witten-Sander model (1981), much research has been conducted on the problem of diffusion-limited aggregation (DLA). In Monte Carlo studies, scaling behaviour has been observed both in this model and in many variants of it. As in critical phenomena, this scaling behaviour is believed to depend on the dimensionality of the lattice but not on the details of its structure (Witten and Sander 1981, 1983, Meakin 1983a, b). Universality is also observed when the condition for adhesion of the diffusing particle to the aggregate is varied (Witten and Sander 1983, Meaking 1983a, b). To be specific, the case where the particle adheres on first contact (absolute adhesion) seems to scale in the same way as in cases where the probability of adhesion on contact is less than one (partial adhesion). At the moment the support for these conjectures for aggregation on Euclidean lattices comes solely from simulations; no exact results are yet available.

In this letter we relate the above two types of universality by presenting an exact bond-to-site transformation from a class of partial adhesion problems to problems with absolute adhesion. The underlying lattice is also transformed. Such mappings have provided important results in the understanding of universality in equilibrium statistical mechanics (Syozi 1972) and percolation (Fisher 1961). We shall illustrate the transformation for both DLA and directed diffusion-limited aggregation (DDLA) on the honeycomb lattice. However, the mapping can be applied to DLA and DDLA on any lattice.

First we review the aggregation process. In both DLA and DDLA a single particle executes a random walk on a lattice containing a static set of particles known as the aggregate. Typically the initial aggregate consists either of a single seed particle or a substrate (i.e., a line of seeds). In DDLA there is a preferred direction for the walk (Meakin 1983c, Jullien *et al* 1984, Nadal *et al* 1984), while in DLA the walk is unbiased. When the diffusing particle reaches a site adjacent to one of the particles in the aggregate, there is some probability that the particle adheres to the aggregate. If this probability is unity, we say there is absolute adhesion; otherwise there is partial adhesion. If the particle adheres to the aggregate, its motion stops, it becomes part of the aggregate, and another diffusing particle is introduced. Otherwise it continues to walk on the unoccupied sites. If the diffusing particle escapes to infinity, another is introduced.

It is necessary to be more precise about what we mean by partial adhesion. In the absence of an aggregate there are several paths that may be taken by the diffusing particle at the next time step. In the presence of the aggregate, some or all of these possible paths may be blocked by particles of the aggregate. We define the discrete variable x (where $0 \leq x \leq 1$) to be the fraction of such possible paths which are blocked. To describe varying adhesion probability, we introduce a one-paramter family, $P_\lambda(x) = x^\lambda$ $(0 \leq \lambda \leq 1)$. Note that $P_0(0) \equiv 0$. The parameter λ is a measure of the adhesion, and the function $P_\lambda(x)$ is the probability that a diffusing particle adheres when a fraction x of its paths are blocked. This class of functions has a number of desirable features. Since $P_\lambda(0) = 0$, if the diffusing particle is not adjacent to the aggregate, there is no chance of adhesion. The property $P_\lambda(1) = 1$ ensures that if all possible paths are blocked, the particle will adhere. Since $P_\lambda(x)$ is an increasing function of x, the adhesion probability increases with the fraction of blocking sites.

Two values of λ deserve special attention. The case $\lambda = 0$ corresponds to absolute adhesion. The diffusing particle checks all possible paths and adheres if any are blocked. The case $\lambda = 1$ $(P_\lambda(x) = x)$ also has a simple interpretation. The diffusing particle selects one of the possible paths and adheres at its present location if that path is blocked. Otherwise it moves along the chosen path. Since $P_\lambda(x)$ is an decreasing function of λ, the intermediate values of λ provide a smooth interpolation between these two cases.

In this letter we present a bond-to-site mapping which transforms partial adhesion problems with $\lambda = 1$ to others with $\lambda = 0$ (absolute adhesion). As well as altering the type of adhesion, the mapping also changes the lattice structure. We explain the transformation on the honeycombe lattice site, first for DDLA and then for DLA. As mentioned before, the mapping is general and can easily be applied to any lattice structure. This correspondence implies an exact equivalence between the scaling behaviours of the two problems. We also examine the scaling behaviour for intermediate values of λ.

We first consider the problem of DDLA on the honeycomb lattice with $\lambda = 1$. The transformed problem is DDLA on a decorated square lattice with absolute adhesion $(\lambda = 0)$. Both lattices are shown in figure 1 (with full lines and full circles). The broken

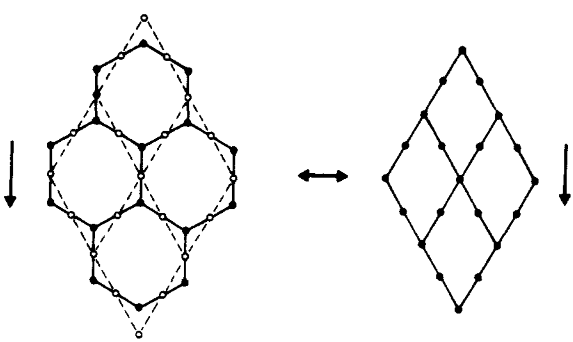

Figure 1. Mapping for DDLA from partial adhesion on the honeycomb lattice to absolute adhesion on the decorated square lattice. The downward arrow indicates the required direction for the walks. The broken lines and open circles show how the transformed lattice is constructed.

lines and open circles will be used later to describe the transformation. The walks on both lattices are directed; in each case upward motion is strictly forbidden. Therefore, at each stage the walk can continue along at most two paths. Choices between paths are made with equal probability.

A site on the honeycomb lattice has either one or two paths beneath it. Therefore, the adhesion can be described as follows. If there is only one path, the presence of an aggregate particle below results in adhesion. If there are two paths, the presence of one aggregate particle results in adhesion with probability $\frac{1}{2}$. If both paths are blocked, the diffusing particle must adhere. In contrast, the adhesion on the decorated square lattice is absolute: if there are two paths downward and one is blocked, the diffusing particle must adhere.

We now describe the mapping between the two problems. The construction of the new lattice (and the allowed walks on this lattice) proceeds by a bond-to-site transformation. On each bond of the old lattice (i.e., the lattice for the ($\lambda = 1$) partial adhesion problem), place a site of the new lattice ($\lambda = 0$). Connect two sites of the new lattice if the corresponding bonds in the old lattice meet at a vertex and there is an allowed walk from one bond to the other in the old lattice. The allowed directions between sites of the new lattice are the same as the directions for the allowed walks between the two bonds of the old lattice. As shown in figure 1, this prescription transforms DDLA on the honeycomb lattice to DDLA on the decorated square lattice. In both lattices each bond can be traversed in only one direction. A detailed explanation of the equivalence will be given shortly.

Note that when the bond-to-site mapping for percolation (Fisher 1961) is applied to bond percolation on the honeycomb lattice, the transformed (or covering) lattice is the Kagomé, not the decorated square. The reason for this is that not all pairs of adjacent sites are connected in our transformed lattice. For example, in figure 2 there is a walk A→C because the walk 1→3→4 is permitted in the old lattice. However, there is no walk A→B since the walk 1→3→2 is not allowed in the old lattice.

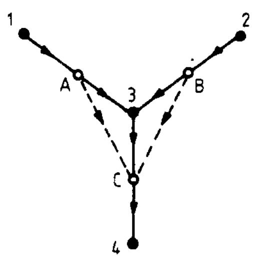

Figure 2. Detail showing allowed directions for walks in DDLA.

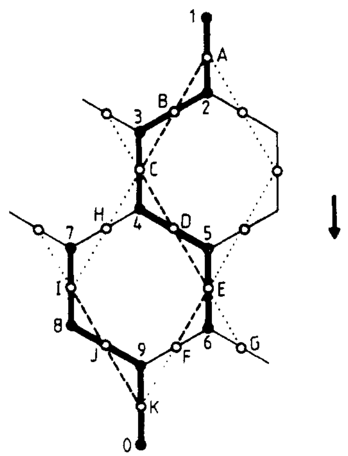

Figure 3. An example of the correspondence between the aggregation processes in DDLA, showing both the aggregates and the walks. On the honeycomb lattice the bold full lines represent the aggregate and walk. On the decorated square lattice these features are represented by broken lines.

To demonstrate the equivalence between the $\lambda = 1$ partial adhesion problem on the honeycomb lattice and the absolute adhesion problem on the decorated square lattice, we first show that there is an exact correspondence between walks on the two lattices even in the presence of the aggregate. In figure 3 we illustrate this by showing the path of a diffusing particle on the honeycomb lattice ($1 \to 2 \to 3 \to 4 \to 5 \to 6$) in the presence of an aggregate (7-8-9-0). The figure also shows the corresponding path for the diffusing particle on the transformed lattice ($A \to B \to C \to D \to E \to F$) and the transformed aggregate (I-J-K).

Away from the aggregate the walks have the same weight since every choice made by one is also made by the other. When the first particle chooses to go from 2 to 3 with probability $\frac{1}{2}$, the second goes from A to B with the same probability. The first particle then advances from 3 to 4 with probability 1, while the second must go from B to C. In the presence of the aggregate the partial adhesion of the first particle corresponds to the choice of the second particle between two paths. When the first particle reaches 4, it chooses not to adhere (to 7) with probability $\frac{1}{2}$, but to move instead to 5. The corresponding particle on the second lattice chooses to go from C to D rather than from C to H (where it would stop), also with probability $\frac{1}{2}$. When the first particle reaches 6, it chooses to adhere (to 9) with probability $\frac{1}{2}$, while the second particle chooses to move from E to F (where it adheres to K) with probability $\frac{1}{2}$.

To see that the adhesion on the transformed lattice is absolute, consider the subsequent evolution of this aggregate. After the adhesion described in the previous paragraph, if a diffusing particle on the honeycomb lattice later reaches 5, it must adhere since the only available path (to 6) is blocked. The particle on the transformed lattice moves to the vacant site E, where it must adhere to the particle at F, although G is unoccupied.

Having demonstrated the equivalence of the walks and the adhesion, we next consider the growth of the aggregates. Again using figure 3, note that on the honeycomb lattice the initial seed at 0 must grow to 9, so the aggregate (0-9) can also be considered as the seed. The corresponding aggregate K is the seed on the decorated square lattice and also has unit probability. Since the walks on the two lattices are equivalent, the subsequent growth of these aggregates will also be in exact correspondence. For every diffusive walk on the honeycomb lattice which misses the aggregate, there is one of equal weight on the decorated square lattice which also escapes. For every walk on the honeycomb lattice which terminates by adhesion at a given site, there is a walk of equal probability on the decorated square lattice which adheres at the corresponding site. Therefore, at each stage the possible aggregates on one lattice can be matched up with equally probable aggregates on the other lattice. Note that this anlaysis also holds if the initial aggregate is a substrate (line of seeds) rather than a single seed. Because of this correspondence the two problems must have the same scaling behaviour.

This correspondence also enables us to relate two conjectured but generally accepted aspects of universality, namely that scaling behaviour is independent of lattice structure and independent of adhesion probability. For example, if we assume that DDLA with absolute adhesion has the same scaling properties on our two lattices, then the mapping shows that the models with $\lambda = 0$ and $\lambda = 1$ on the honeycomb lattice have the same exponents. In particular, the average height L of the aggregates has the asymptotic form $L \sim N^{\nu_\parallel}$ (Nadal *et al* 1984), where N is the number of particles in the aggregate. The exponent $\nu_\parallel$ must be the same in both problems.

In fact, the intermediate cases with $0 < \lambda < 1$ also have this value of $\nu_\parallel$. To see this, recall that the adhesion probability $P_\lambda(x)$ is a decreasing function of λ. The average height $L(\lambda)$ is therefore a decreasing function of λ—the larger the adhesion probability, the greater the average height. Thus, $L(\lambda)$ is bounded above and below by $L(0)$ and $L(1)$, and these both grow as $N^{\nu_\parallel}$. We conclude that $L(\lambda) \sim N^{\nu_\parallel}$ for $0 \leq \lambda \leq 1$. Therefore, if lattice structure is irrelevant, the adhesion probability is also irrelevant. Note that this argument does not depend on the particular one-parameter family we have chosen. Any intermediate case with $P(0) = 0$ and $x \leq P(x) \leq 1$ will be bounded above and below in the same fashion.

The converse is also true. If we assume that the scaling behaviour on the honeycomb lattice is independent of adhesion probability, then the mapping proves that DDLA with absolute adhesion has the same behaviour on both the honeycomb and decorated square lattices. Therefore, the mapping relates the two conjectures that adhesion probability and lattice structure are irrelevant.

In one case this result can be further strengthened. DDLA on the Bethe lattice (Bradley and Strenski 1984a, b) with partial adhesion ($\lambda = 1$) maps to DDLA on the same lattice with absolute adhesion under this transformation. No assumption is needed to prove that the adhesion probability is irrelevant.

The mapping is not restricted to directed diffusion. However, in DLA the transformed walks are more complicated. To illustrate this point, we consider DLA on the honeycomb lattice with partial adhesion ($\lambda = 1$). In this problem the diffusing particle chooses one of three directions with probability $\frac{1}{3}$, and stops at its present location if the site in that direction is already occupied. Thus, with one nearest-neighbour site occupied the diffusing particle stops with probability $\frac{1}{3}$, and with two nearest-neighbour sites occupied it stops with probability $\frac{2}{3}$.

When the prescription given before is applied to this problem, the transformed lattice is the Kagomé lattice (figure 4) with absolute adhesion ($\lambda = 0$). Note that now the bonds may be traversed in either direction. However, there are some restrictions on the diffusive walk on this lattice. Although a given site on the Kagomé lattice has four paths leading from it, not all of these will be allowed walks at any one time step. The allowed directions will depend on the direction of the motion at the previous time step. For example, suppose the particle on the Kagomé lattice (figure 5) is at B. If the last walk on the honeycomb lattice was $3 \to 4$, then the allowed paths at the next time step are $B \to C$ and $B \to E$. However, if the previous motion was $4 \to 3$, then the allowed paths are instead $B \to A$ and $B \to D$. Since the motion $3 \to 4 \to 3$ is permitted,

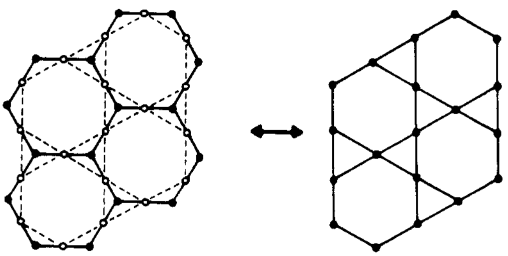

Figure 4. Mapping for DLA from partial adhesion on the honeycomb lattice to absolute adhesion on the Kagomé lattice. The broken lines and open circles show how the transformed lattice is constructed.

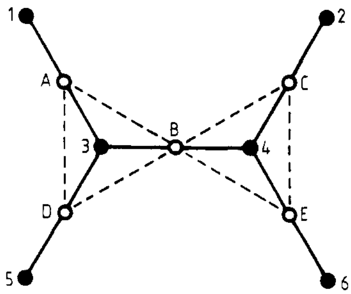

Figure 5. Detail showing restrictions on transformed walks in DLA.

the particle on the Kagomé lattice may also remain at B at a given time step. If this occurs then the allowed directions for the next step in the walk will change. These conditions on the transformed walk can be summarised by ascribing a binary 'momentum' to the transformed particle. This 'momentum' lies along the direction of the bond in the old lattice. At each stage of the walk on the Kagomé lattice the particle can move in one of the two directions along its present 'momentum', or this 'momentum' can flip while the position remains constant. Each of these possibilities has probability $\frac{1}{3}$. The resulting walk is still unbiased globally, but has a preference locally for motion in the same direction.

In summary, we have constructed a general mapping for aggregation from a particular form of partial adhesion on one lattice to others with absolute adhesion on a transformed lattice. Examples of the transformation have been presented on the honeycomb lattice for both DDLA and DLA. The mapping relates two conjectured aspects of universality, namely that lattice structure and adhesion probability are irrelevant.

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References

Bradley R M and Strenski P N 1984a to appear in *Phys. Rev.* B (*Rapid Communications*)
—— 1984b *Directed Aggregation on the Bethe Lattice: Scaling, Universality and Mappings, preprint*
Fisher M E 1961 *J. Math. Phys.* **2** 620-7
Jullien R, Kolb M and Botet R 1984 *J. Physique* **45** 395-9
Meaking P 1983a *Phys. Rev.* A **27** 604-7
—— 1983b *Phys. Rev.* A **27** 1495-507
—— 1983c *Phys. Rev.* B **28** 5221-4
Nadal J P, Derrida B and Vannimenus J 1984 *Phys. Rev.* B **30** 376-83
Syozi I 1972 in *Phase Transitions and Critical Phenomena* ed C Domb and M S Green (New York: Academic) vol 1, pp 270-329
Witten T A and Sander L M 1981 *Phys. Rev. Lett.* **47** 1400-3
—— 1983 *Phys. Rev.* B **27** 5686-97